Low-frequency hydromagnetic modes of a uniformly magnetized liquid star

S. I. Bastrukov and D. V. Podgainy
Computer Physics Division, Laboratory of Computing Technique and Automation, Joint Institute for Nuclear Research, 141980 Dubna, Russia

(Received 29 February 1996)

The low-frequency hydromagnetic vibrations of a star are studied in the homogeneous model of a spherical mass of incompressible inviscid fluid with a uniform magnetic field inside. Taking into consideration the fact that the presence of the magnetic field inside liquid imparts to it properties of an elastic substance (allowing propagation of the transverse magnetohydrodynamic wave), it is argued that the eigenmodes of hydromagnetic vibrations can be specified in the same manner as eigenmodes of elastodynamic vibrations. An explicit form is obtained for the frequency of the poloidal and toroidal hydromagnetic vibrations. Numerical estimates are presented of the frequency, computed in the homogeneous model with parameters typical of a neutron star. [S1063-651X(96)09008-3]

PACS number(s): 52.35.Bj, 04.40.−b

It has been known for a long time that the presence of magnetic field inside a star produces the same effect as a rigid-body rotation, that is, it tends to flatten the star by contracting magnetic poles in the direction of the field—the Chandrasekhar-Fermi effect [1]. For a homogeneous self-gravitating liquid sphere of the radius R and mass M, with the uniform magnetic field of intensity B inside, the ellipticity of an oblate configuration is evaluated as $\epsilon \sim E_{\rm mag}/E_{\rm gr}$, where $E_{\rm mag}\sim B^2R^3$ and $E_{\rm gr}\sim GM^2/R$ are the magnetic and gravitational energy, respectively [2]. However, from this estimate it follows that the *static* effect of magnetic flattening turns out to be negligibly small for the stars from the main sequence, in which $B\sim 10-10^3$ G [3,4], as well as for strongly magnetized degenerated compact objects like neutron stars, in which $B\sim 10^{11}-10^{13}$ G [5]. This is one of the main reasons why the investigations on stellar magnetism are dominated by the search for *dynamical* manifestation of the presence of magnetic fields in the interior of stars. In this Brief Report one problem from this area is considered. Specifically, we focus on the eigenmodes of hydromagnetic vibrations of a star modeled by a uniformly magnetized homogeneous liquid sphere. While the model of uniformly distributed matter does not reflect the realistic density profile of the known stellar classes, the physical significance of homogeneous models is that they allow one to gain a clear impression of stellar normal modes and to elucidate the connection between different kinds of energy stored in the star and its electromagnetic activity.

In the homogeneous model under consideration a magnetic star is thought of as a heavy spherical mass of nonviscous incompressible liquid with the uniform magnetic field. The electrical conductivity of stellar liquid is assumed to be infinitely large. It is well established that the behavior of this liquid is adequately described by equations of the magnetohydrodynamics [4,6]:

$$\mathrm{div}\mathbf{H}=0,\quad \frac{\partial \mathbf{H}}{\partial t}=\mathbf{rot}[\mathbf{V}\times\mathbf{H}], \tag{1}$$

$$\frac{d\rho}{dt}+\rho\,\mathrm{div}\mathbf{V}=0, \tag{2}$$

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla W+\frac{M}{4\pi}(\mathbf{H}\nabla)\mathbf{H},\quad W=P+\frac{\mu}{8\pi}H^2. \tag{3}$$

Here ρ and $\mathbf{V}$ are the average density and mean velocity. $\mathbf{H}$ stands for the intensity of magnetic field, μ is the magnetic permeability ($\mathbf{B}=\mu\mathbf{H}$), and W is hydromagnetic pressure (d/dt denotes the convective derivative). The distinguishing feature of the dynamical behavior of a liquid governed by the above equations is that it permits propagation of the transverse wave of Alfvén (along with the longitudinal sound wave). For incompressible fluid, the linearized equations containing solution which represent the transmission of a hydromagnetic wave can be written as follows:

$$\frac{\partial \delta V_i}{\partial x_i}=0,\quad \frac{\partial \delta H_k}{\partial x_k}=0, \tag{4}$$

$$\rho\frac{\partial \delta V_i}{\partial t}-\frac{\mu H_k}{4\pi}\frac{\partial \delta H_i}{\partial x_k}=0, \tag{5}$$

$$\frac{\partial \delta H_i}{\partial t}-H_k\frac{\partial \delta V_i}{\partial x_k}=0, \tag{6}$$

where δV_k and δH_k are the components of fluctuating velocity and intensity of the magnetic field. In deriving Eqs. (4)–(6), the trivial solution of the Laplace equation $\Delta\,\delta W=0$ for fluctuations in hydromagnetic pressure has been used: $\delta W=0$ (see, for details, Ref. [6], Chap. IV, Sec. 39, pp. 155 and 156). This is the case when gravitational vibrations are not excited and the hydromagnetic wave is the only degree of activity of the magnetized stellar matter governed by the magnetofluid-dynamical equations (1)–(3). Taking the time and space dependence of the fluctuating variables δV_k and δH_k in the plane-wave form $e^{i(\mathbf{kr}-\omega t)}$, from Eqs. (4)–(6) it can be immediately verified that the hydromagnetic wave propagates with the phase velocity

$$c^2=\omega^2/k^2=V_A^2\cos^2\theta,\quad V_A^2=\frac{\mu H^2}{4\pi\rho}, \tag{7}$$

where V_A is the Alfvén velocity and θ is the inclination of the direction of wave propagation to the direction of **H**. From Eqs. (4)–(6) it follows that the energy balance of the hydromagnetic wave's process is controlled by the equation [6]

$$\frac{1}{2}\rho(\delta\mathbf{V})^2=\frac{\mu}{8\pi}(\delta\mathbf{H})^2, \tag{8}$$

that is, the mean energy of the wave in the kinetic motion of liquid equals mean energy of the wave in the magnetic field. In magnetohydrodynamics (see, for instance, [4,6]) it is stressed that the Alfvén wave follows the magnetic lines of force and the physical nature of this wave is analogous to the transverse wave propagating along the elastic string (the magnetic line of force behaves like a stretched string frozen in liquid). Thus, the presence of the homogeneous magnetic field inside a liquid imparts to it the dynamical properties of an elastic substance in the sense that propagation of the undamped *transverse* vibrations is the feature inherent in elastic solid.

Taking into consideration this similarity in the behavior of magnetized liquid and elastic solid in bulk, it seems reasonable to assume that undamped hydromagnetic vibrations of a magnetized liquid drop are developed in a manner of eigenvibrations of an elastic globe. The eigenmodes of an elastic star have recently been studied in Ref. [7], associated with the spheroidal (poloidal mode) and torsional (toroidal mode) gravitation-elastic vibrations. Based on this assumption, it is argued below that the low-frequency hydromagnetic eigenmodes of a uniformly magnetized star can be specified as the poloidal and toroidal ones (depending upon the vector or pseudovector nature of excited solenoidal flow) in accord with the elastodynamic classification of normal modes adopted in Ref. [7].

To calculate fundamental frequencies of the volume hydromagnetic oscillations we take advantage of the general variational principle [6]. The procedure is the following. Scalar multiplication of Eq. (5) by δV_i and integration over the star volume (on the star surface it is assumed that $\delta\mathbf{H}|_{r=R}=0$) leads to the integral equation of energy balance

$$\frac{\partial}{\partial t}\int_V\frac{\rho\,\delta V^2}{2}d\tau-\frac{\mu}{4\pi}\int_V\delta V_iH_k\frac{\partial\,\delta H_i}{\partial x_k}d\tau=0. \tag{9}$$

It is convenient to represent the small-amplitude deviations in the velocity and magnetic intensity as follows:

$$\delta V_i=a_i^L(\mathbf{r})\dot{\alpha}_L(t),\quad \delta H_i=h_i^L(\mathbf{r})\alpha_L(t), \tag{10}$$

where L is the multipole order of vibration. Inserting (10) into (6), one finds

$$h_i^L=H_k\frac{\partial a_i^L}{\partial x_k}. \tag{11}$$

In the spherical coordinates, Eq. (11) is equivalent to

$$h_r=\left[H_r\frac{\partial}{\partial r}+\frac{H_\theta}{r}\frac{\partial}{\partial\theta}+\frac{H_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]a_r-\frac{H_\theta a_\theta+H_\phi a_\phi}{r},$$

$$h_\theta=\left[H_r\frac{\partial}{\partial r}+\frac{H_\theta}{r}\frac{\partial}{\partial\theta}+\frac{H_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]a_\theta+\frac{H_\theta a_r-H_\phi a_\phi\mathrm{ctg}\theta}{r},$$

$$h_\phi=\left[H_r\frac{\partial}{\partial r}+\frac{H_\theta}{r}\frac{\partial}{\partial\theta}+\frac{H_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]a_\phi+\frac{H_\phi a_r+H_\phi a_\theta\mathrm{ctg}\theta}{r}.$$

Substitution of (10) into Eq. (9) transforms this latter into an equation for the Hamiltonian of normal vibrations

$$\frac{dH}{dt}=0,\quad H=\frac{1}{2}M_L\dot{\alpha}_L^2+\frac{1}{2}K_L\alpha_L^2, \tag{12}$$

where the inertia M_L and the stiffness K_L are given by

$$M_L=\int_V\rho a_i^La_i^Ld\tau,\quad K_L=\frac{\mu}{4\pi}\int_V h_i^Lh_i^Ld\tau. \tag{13}$$

Thus, to compute the eigenfrequency, $\omega^2=K_L/M_L$, of hydromagnetic vibrations, one has to specify the velocity field of the oscillating solenoidal flow.

Poloidal hydromagnetic modes. Following Ref. [1] we restrict our consideration to the homogeneous model of a star inside which the uniform magnetic field of intensity **H** is directed along the polar z axis,

$$H_r=\eta H,\quad H_\theta=-(1-\eta^2)^{1/2}H,\quad H_\phi=0,\quad \eta=\cos\theta. \tag{14}$$

It can be straightforwardly verified that one of the hydromagnetic eigenmodes of the Hamiltonian (12) is associated with excitation of the poloidal field of velocity,

$$\delta\mathbf{V}=\frac{A_L}{L+1}\mathrm{rot\ rot}[\mathbf{r}r^LP_L(\eta)]\dot{\alpha}_L(t)$$

$$=A_L\mathrm{grad}[r^LP_L(\eta)]\dot{\alpha}_L(t),\quad A_L=\frac{1}{LR^{L-2}}. \tag{15}$$

Here $P_L(\eta)$ is the Legendre polynomial. The field (15) provides description of spheroidal vibrations of an elastic star [7]. The components of instantaneous poloidal displacements are written as follows [1]:

$$a_r^L=\frac{r^{L-1}}{R^{L-2}}P_L(\eta),$$

$$a_\theta^L=-\frac{r^{L-1}}{LR^{L-2}}(1-\eta^2)^{1/2}\frac{\partial P_L(\eta)}{\partial\eta},$$

$$a_\phi^L=0. \tag{16}$$

The deviation in the intensity of the magnetic field is given by [1]

$$h_r=H(L-1)\frac{r^{L-2}}{R^{L-2}}P_{L-1}(\eta),$$

$$h_\theta=-H\frac{r^{L-2}}{R^{L-2}}(1-\eta^2)^{1/2}\frac{\partial P_{L-1}(\eta)}{\partial\eta},$$

$$h_\phi=0. \tag{17}$$

The mass parameter computed with the field (16) equals

$$M_L = \frac{4\pi\rho R^5}{L(2L+1)}. \tag{18}$$

For the stiffness, we obtain

$$K_L = \mu H^2 R^3 \frac{L-1}{2L-1}. \tag{19}$$

From (19) it follows that a monopole mode cannot exist. This is the consequence of incompressibility. The dipole poloidal field of velocity, as it follows from the Hamiltonian (12), contributes to the kinetic energy, whereas the potential energy vanishes. The disturbance of the dipole poloidal velocity can result in the center-of-mass motion, without changing the intrinsic state of the star. The eigenfrequency of the poloidal hydromagnetic vibrations is given by

$$\omega_{\mathrm{p}}^2 = \Omega_A^2 L(L-1)\frac{2L+1}{2L-1}, \tag{20}$$

where

$$\Omega_A^2 = \frac{V_A^2}{R^2} = \frac{\mu H^2}{4\pi\rho R^2} \tag{21}$$

is the Alfvén frequency.

Toroidal hydromagnetic modes. The property of elasticity of magnetized liquid allows one to consider shear hydromagnetic oscillations. In the elastic sphere, these excitations are described by the toroidal field of velocity [7]:

$$\delta\mathbf{V} = A_L \mathrm{rot}[\mathbf{r} r^L P_L(\eta)]\dot{\alpha}_L(t) = [\mathbf{\Omega}(\mathbf{r},t)\times\mathbf{r}], \tag{22}$$

where

$$\mathbf{\Omega}(\mathbf{r},t) = A_L \mathrm{grad}[r^L P_L(\eta)]\dot{\alpha}_L(t), \quad A_L = \frac{1}{R^{L-1}} \tag{23}$$

is the frequency of local torsional oscillations. The components of toroidal instantaneous displacements are written as

$$a_r = 0, \quad a_\theta = 0, \quad a_\phi = -\frac{r^L}{R^{L-1}}(1-\eta^2)^{1/2}\frac{\partial P_L(\eta)}{\partial\eta}, \tag{24}$$

and the corresponding fluctuations in the intensity of magnetic field are described by

$$h_r = 0, \;\; h_\theta = 0, \;\; h_\phi = H(L+1)\frac{r^{L-1}}{R^{L-1}}(1-\eta^2)^{1/2}\frac{\partial P_{L-1}(\eta)}{\partial\eta}. \tag{25}$$

The mass parameter (torsional moment of inertia) computed with the toroidal field (24) is given by

$$M_L = 4\pi\rho R^5 \frac{L(L+1)}{(2L+1)(2L+3)}. \tag{26}$$

For the stiffness, we obtain

$$K_L = \mu H^2 R^3 \frac{L(L-1)(L+1)^2}{(2L+1)(2L-1)}. \tag{27}$$

From (27) it follows that the dipole toroidal mode is not an eigenmode of the oscillator Hamiltonian (12). Excitation of the toroidal dipole displacements can lead to the rigid-body rotation of the liquid sphere. The eigenfrequency of the toroidal hydromagnetic vibrations is given by

$$\omega_{\mathrm{t}}^2 = \Omega_A^2 (L^2-1)\frac{2L+3}{2L-1}, \tag{28}$$

where Ω_A is the Alfvén frequency defined by (21).

Equations (20) and (28) are the basic predictions of the homogeneous model considered. From these equations it follows that the eigenfrequency of hydromagnetic modes is proportional to the intensity of magnetic field H and depends upon the star size as $\sim 1/R$. The period P_{hm} of both the poloidal and toroidal hydromagnetic vibrations monotonically decreases with increasing the multipole order L as $P_{\mathrm{hm}} \sim 1/L$.

As a representative example we apply the above homogeneous model to a magnetic star with parameters typical of a neutron star. Among known stellar classes, the neutron stars possess the strongest concentration of magnetic energy. An extensive discussion of the origin and evolution of the magnetic field in degenerated stars can be found in Ref. [5]. We consider a model of a star with mass $1.0 < M < 1.4[M_\odot]$ [the average density $\rho \sim (2-3)\rho_N$, where $\rho_N = 2.8\times 10^{14}$ g/cm^3 is the normal nuclear density], radius $0.8 < R < 12$ (km), and with a magnetic field of intensity $B \sim 10^{13}$ G ($\mu \sim 1$). From Eqs. (20) and (28), it follows that the Alfvén frequency is given by $\Omega_A \sim 10^{-1}$ Hz. Notice that with the above set of parameters the frequency of hydromagnetic vibrations $\omega_{\mathrm{hm}} \sim \sqrt{B^2/\rho R^2} \sim 10^{-2} - 10^{-1}$ Hz is much less than that of gravitational nonradial vibrations. The latter, computed in the homogeneous model of a self-gravitating liquid sphere, is estimated to be $\omega_{\mathrm{gr}} \sim \sqrt{G\rho} \sim 10^3 - 10^4$ Hz [8,9]. So the large difference emphasizes the fact that the excitation of gravitational vibrations in the superdense self-gravitating matter requires much more energy than that required for hydromagnetic vibrations.

It is remarkable that the computed period of hydromagnetic vibrations $P_{\mathrm{hm}} = 2\pi/\omega_{\mathrm{hm}} \sim 0.1 - 10$ s (for L from the interval $2 < L < 20$) is of the same order as the period of radio emission of pulsars (see the systematic data presented in Fig. 5 of Ref. [5]). While the model considered does not disclose the mechanism governing the electromagnetic activity of a neutron star, the above correspondence of periods may be interpreted in favor of the hypothesis, advanced long ago in Ref. [10], that the magnetic energy stored in the newly born neutron star can be released by means of transformation of the energy of hydromagnetic vibrations into the energy of electromagnetic radiation.

[1] S. Chandrasekhar and E. Fermi, Asrophys. J. **118**, 116 (1953).
[2] V.C.A. Ferraro, Astrophys. J. **120**, 407 (1954).
[3] E.N. Parker, *Cosmical Magnetic Fields* (Clarendon, Oxford, 1979).
[4] E.R. Priest, *Solar Magnetohydrodynamics* (Reidel, Dordrecht, 1982).
[5] G. Chanmugam, Annu. Rev. Astron. Astrophys. **30**, 143 (1992).
[6] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961).
[7] S.I. Bastrukov, Phys. Rev. E **53**, 1917 (1996).
[8] S.I. Bastrukov, I.V. Molodtsova, and A.A. Bukatina, Astrophys. **38**, 123 (1995).
[9] S.I. Bastrukov, I.V. Molodtsova, V.V. Papoyan, and F. Weber, J. Phys. G **22**, L33 (1996).
[10] F. Hoyle, J.V. Narlikar, and J.A. Wheeler, Nature **203**, 914 (1964).